Chapter 5: Application of Definite Integration

EXERCISE 5.1 [PAGE 187]

Exercise 5.1 | Q 1.1 | Page 187

Find the area of the region bounded by the following curves, X-axis and the given lines:
y = 2x, x = 0, x = 5

SOLUTION

Required area = $\int_0^5 y \cdot dx, \text{where } y = 2x$

$= \int_0^5 2x \cdot dx$

$= \left[\frac{2x^2}{2}\right]_0^5$

= 25 – 0

= 25 sq units.

Exercise 5.1 | Q 1.2 | Page 187

Find the area of the region bounded by the following curves, X-axis and the given lines:
x = 2y, y = 0, y = 4

SOLUTION

Required area = $\int_0^4 x \cdot dy, \text{where } x = 2y$

$= \int_0^4 2y \cdot dy$

$= \left[\frac{2y^2}{2}\right]_0^4$

= 16 – 0

= 16 sq units.

Exercise 5.1 | Q 1.3 | Page 187

Find the area of the region bounded by the following curves, X-axis and the given lines :
x = 0, x = 5, y = 0, y = 4

SOLUTION

Required area = $\int_0^5 y \cdot dx, \text{where } y = 4$

$= \int_0^5 4 \cdot dx$

$= [4x]_0^5$

= 20 – 0

= 20 sq units.

Exercise 5.1 | Q 1.4 | Page 187

Find the area of the region bounded by the following curves, X-axis and the given lines :
y = sin x, x = 0, x = π/2

SOLUTION

Required area = $\int_0^{\frac{\pi}{2}} y \cdot dx,\ \text{where } y = \sin x$

$= \int_0^{\frac{\pi}{2}} \sin x \cdot dx$

$= [-\cos x]_0^{\frac{\pi}{2}}$

$= -\cos\ \frac{\pi}{2} + \cos 0$

= 0 + 1

= 1 sq unit.

Exercise 5.1 | Q 1.5 | Page 187

Find the area of the region bounded by the following curves, X-axis and the given lines:
xy = 2, x = 1, x = 4

SOLUTION

For xy = 2, y = $\frac{2}{x}$.

Required area = $\int_1^4 y \cdot dx, \text{ where } y = \frac{2}{x}$

= $\int_1^4 \frac{2}{x} \cdot dx$

= $[2\log|x|]_1^4$

= 2 log 4 – 2 log 1

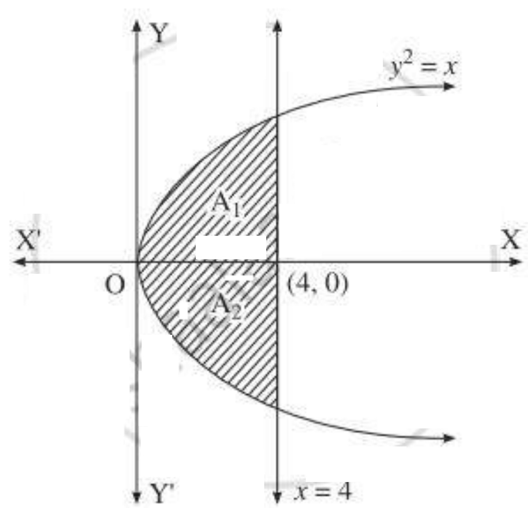
= 2 log 4 – 0

= 2 log 4 sq units.

Exercise 5.1 | Q 1.6 | Page 187

Find the area of the region bounded by the following curves, X-axis and the given lines :
$y^2 = x$, x = 0, x = 4

SOLUTION



The required area consists of two bounded regions A_1 and A_2 which are equal in areas.

For $y^2 = x$, $y = \sqrt{x}$

Required area = $A_1 + A_2 = 2A_1$

$$= 2\int_0^4 y \cdot dx, \text{ where } y = \sqrt{x}$$

$$= 2\int_0^4 \sqrt{x} \cdot dx$$

$$= 2\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$$

$$= 2\left[\frac{2}{3}(4)^{\frac{3}{2}} - 0\right]$$

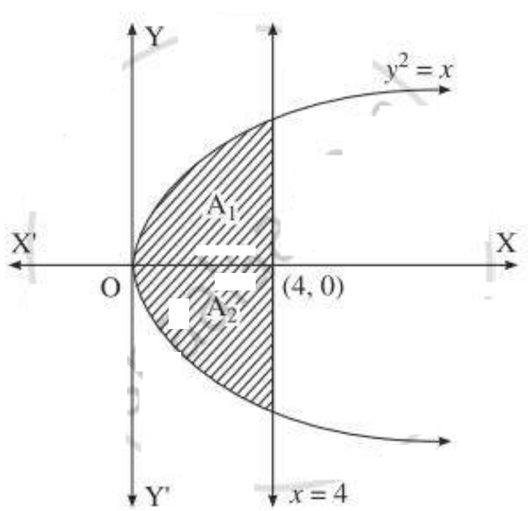
$$= 2\left[\frac{2}{3}\left(2^2\right)^{\frac{3}{2}}\right]$$

$$= \frac{32}{3} \text{sq units.}$$

Exercise 5.1 | Q 1.7 | Page 187

Find the area of the region bounded by the following curves, X-axis and the given lines:
$y^2 = 16x$, $x = 0$, $x = 4$

SOLUTION



The required area consists of two bounded regions A_1 and A_2 which are equal in areas.

For $y^2 = x$, $y = \sqrt{x}$

Required area = $A_1 + A_2 = 2A_1$

$$= 2\int_0^4 y \cdot dx, \text{ where } y = \sqrt{x}$$

$$= 2\int_0^4 \sqrt{x} \cdot dx$$

$$= 2\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$$

$$= 2\left[\frac{2}{3}(4)^{\frac{3}{2}} - 0\right]$$

$$= 2\left[\frac{2}{3}\left(2^2\right)^{\frac{3}{2}}\right]$$

$$= \frac{128}{3} \text{ sq units.}$$

Exercise 5.1 | Q 2.1 | Page 187

Find the area of the region bounded by the parabola: $y^2 = 16x$ and its latus rectum.

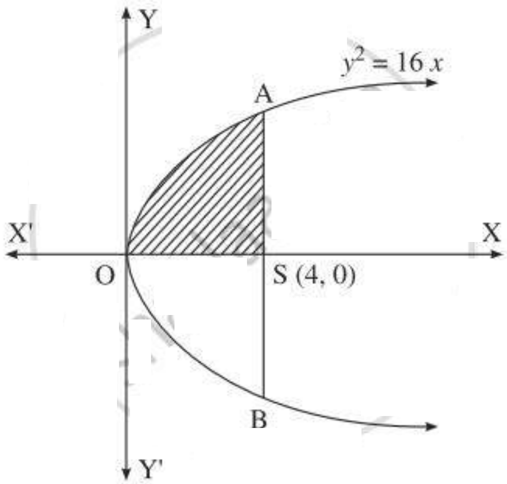
SOLUTION

Comparing $y^2 = 16x$ with $y^2 = 4ax$, we get

$4a = 16$

$\therefore a = 4$

$\therefore$ focus is $S(a, 0) = (4, 0)$



For $y^2 = 16x$, $y = 4\sqrt{x}$

Required area = area of the region OBSAO

= 2[area of the region OSAO]

$= 2\int_0^4 y \cdot dx,\ \text{where } y = 4\sqrt{x}$

$= 2\int_0^4 4\sqrt{x} \cdot dx$

$= 8\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$

$= 8\left[\frac{2}{3}(4)^{\frac{3}{2}} - 0\right]$

$= 8\left[\frac{2}{3}\left(2^2\right)^{\frac{3}{2}}\right]$

$= \frac{128}{3}$ sq units.

Exercise 5.1 | Q 2.2 | Page 187

Find the area of the region bounded by the parabola: $y = 4 - x^2$ and the X-axis.

SOLUTION

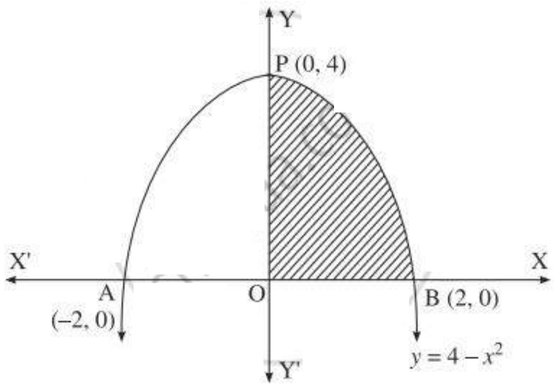
The equation of the parabola is $y = 4 - x^2$

$\therefore x^2 = 4 - y$, i.e. $(x - 0)^2 = -(y - 4)$

It has vertex at P(0, 4)
For points of intersection of the parabola with X-axis,
we put y = 0 in its equation.

∴ $0 = 4 - x^2$
∴ $x^2 = 4$
∴ $x = \pm 2$.

∴ the parabola intersect the X-axis at A (– 2, 0) and B(2, 0)



Required area = area of he region APBOA
= 2[area of the region OPBO]

$$= 2\int y \cdot dx, \text{where } y = 4 - x^2$$

$$= 2\int_0^2 (4 - x^2) \cdot dx$$

$$= 8\int_0^2 1 \cdot dx - 2\int_0^2 x^2 \cdot dx$$

$$= 8[x]_0^2 - 2\left[\frac{x^3}{3}\right]_0^2$$

$$= 8(2 - 0) - \frac{2}{3}(8 - 0)$$

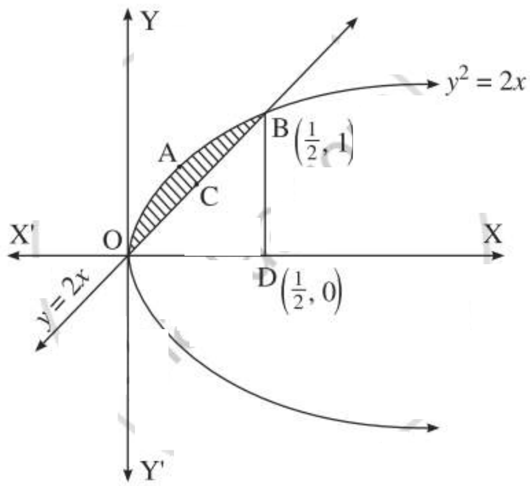
$$= 16 - \frac{16}{3}$$

$$= \frac{32}{3} \text{sq units.}$$

Exercise 5.1 | Q 3.1 | Page 187
Find the area of the region included between: $y^2 = 2x$ and $y = 2x$

SOLUTION

The vertex of the parabola $y^2 = 2x$ is at the origin O = (0, 0).



To find the points of intersection of the line and the parabola, equaling the values of 2x from both the equations we get,
$\therefore y^2 = y$
$\therefore y^2 - y = 0$
$\therefore y(y - 1) = 0$
$\therefore y = 0$ or $y = 1$

When $y = 0$, $x = \dfrac{0}{2} = 0$

When $y = 1$, $x = \dfrac{1}{2}$

$\therefore$ the points of intersection are O(0, 0) and $B\left(\dfrac{1}{2}, 1\right)$

Required area = area of the region OABCO
= area of the region OABDO – area of the region OCBDO

Now, area of the region OABDO
= area under the parabola $y^2 = 2x$ between $x = 0$ and $x = \dfrac{1}{2}$

$= \int_0^{\frac{1}{2}} y \cdot dx$, where $y = \sqrt{2x}$

$= \int_0^{\frac{1}{2}} \sqrt{2x}dx$

$= \sqrt{2}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{\frac{1}{2}}$

$= \sqrt{2}\left[\frac{2}{3}\left(\frac{1}{2}\right)^{\frac{3}{2}} - 0\right]$

$= \sqrt{2}\left[\frac{2}{3} \cdot \frac{1}{2\sqrt{2}}\right]$

$= \frac{1}{3}$

Area of the region OCBDO

= area under the line y

= 2x between x

= 0 and x = $\frac{1}{2}$

$= \int_0^{\frac{1}{2}} y \cdot dx$, where $y = 2x$

$= \int_0^{\frac{1}{2}} 2x \cdot dx$

$= \left[\frac{2x^2}{2}\right]_0^{\frac{1}{2}}$

$= \frac{1}{4} - 0$

$= \frac{1}{4}$

∴ required area

$= \frac{1}{3} = \frac{1}{4}$

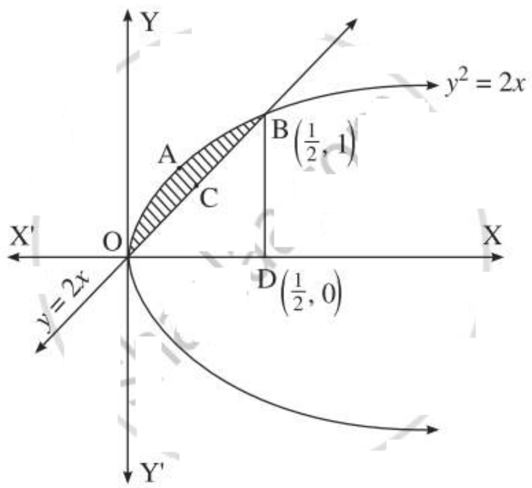
$= \frac{1}{12}$ sq unit.

Exercise 5.1 | Q 3.2 | Page 187

Find the area of the region included between: $y^2 = 4x$, and $y = x$

SOLUTION

The vertex of the parabola $y^2 = 4x$ is at the origin O = (0, 0).



To find the points of intersection of the line and the parabola, equaling the values of 4x from both the equations we get,

∴ $y^2 = y$

∴ $y^2 - y = 0$

∴ $y(y - 1) = 0$

∴ $y = 0$ or $y = 1$

When y = 0, x = $\frac{0}{2}$ = 0

When y = 1, x = $\frac{1}{2}$

∴ the points of intersection are O(0, 0) and $\mathrm{B}\left(\frac{1}{2}, 1\right)$

Required area = area of the region OABCO

= area of the region OABDO – area of the region OCBDO

Now, area of the region OABDO

= area under the parabola y^2 = 4x between x = 0 and x = $\frac{1}{2}$

= $\int_0^{\frac{1}{2}} y \cdot dx, \text{ where } y = \sqrt{x}x$

= $\int_0^{\frac{1}{2}} \sqrt{2}x dx$

= $\sqrt{2}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{\frac{1}{2}}$

= $\sqrt{2}\left[\frac{2}{3}\left(\frac{1}{2}\right)^{\frac{3}{2}} - 0\right]$

= $\sqrt{2}\left[\frac{2}{3} \cdot \frac{1}{2\sqrt{2}}\right]$

= $\frac{1}{3}$

Area of the region OCBDO

= area under the line y

= 2x between x

= 0 and x = $\frac{1}{2}$

$$= \int_0^{\frac{1}{2}} y \cdot dx, \text{where } y = x$$

$$= \int_0^{\frac{1}{2}} 2x \cdot dx$$

$$= \left[\frac{2x^2}{2}\right]_0^{\frac{1}{2}}$$

$$= \frac{4}{1} - 0$$

$$= \frac{4}{3}$$

$\therefore$ required area

$$= \frac{4}{1} = \frac{4}{3}$$

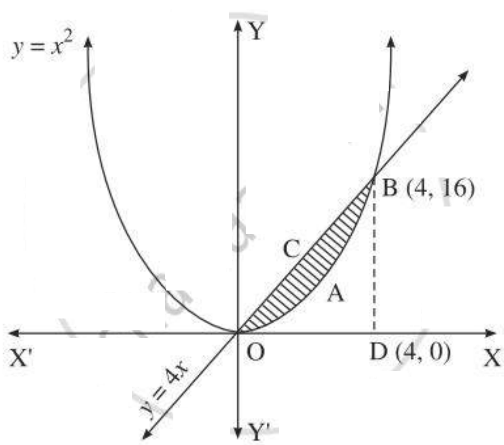
$$= \frac{8}{3}\text{sq units.}$$

Exercise 5.1 | Q 3.3 | Page 187

Find the area of the region included between: $y = x^2$ and the line $y = 4x$

SOLUTION

The vertex of the parabola $y = x^2$ is at the origin O(0, 0)
To find the points of the intersection of the line and the parabola.



Equating the values of y from the two equations, we get

$x^2 = 4x$
$\therefore x^2 - 4x = 0$
$\therefore x(x - 4) = 0$
$\therefore x = 0, x = 4$
When $x = 0$, $y = 4(0) = 0$
When $x = 4$, $y = 4(4) = 16$

$\therefore$ the points of intersection are O(0, 0) and B(4, 16)
Required area = area of the region OABCO

= (area of the region ODBCO) – (area of the region ODBAO)
Now, area of the region ODBCO

= area under the line $y = 4x$ between $x = 0$ and $x = 4$

$$= \int_0^4 y \cdot dx, \text{ where } y = 4x$$

$$= \int_0^4 4x \cdot dx$$

$$= 4\int_0^4 x \cdot dx$$

$$= 4\left[\frac{x^2}{2}\right]_0^4$$

$= 2(16 - 0)$

$= 32$

Area of the region ODBAO

= area under the parabola $y = x^2$ between $x = 0$ and $x = 4$

$$= \int_0^4 y \cdot dx, \text{ where } y = x^2$$

$$= \int_0^4 x^2 \cdot dx$$

$= \left[\frac{x^3}{3}\right]_0^4$

$= \frac{1}{3}(64 - 0)$

$= \frac{64}{3}$

$\therefore$ required area

$= 32 - \frac{64}{3}$

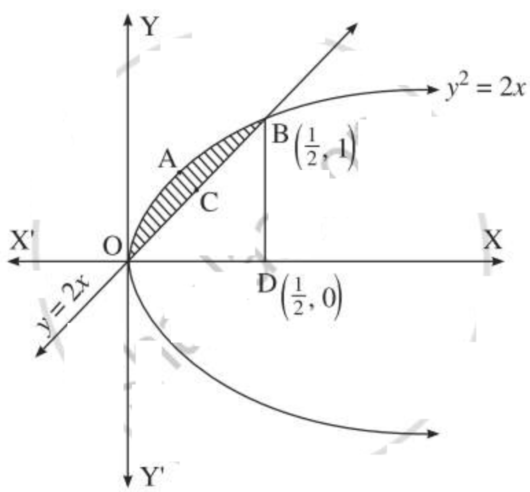
$= \frac{32}{3}$ sq units.

Exercise 5.1 | Q 3.4 | Page 187

Find the area of the region included between: $y^2 = 4ax$ and the line $y = x$

SOLUTION

The vertex of the parabola $y^2 = 4ax$ is at the origin O = (0, 0).



To find the points of intersection of the line and the parabola, equaling the values of 4ax from both the equations we get,

$\therefore y^2 = y$

$\therefore y^2 - y = 0$

$\therefore y(y - 1) = 0$

$\therefore y = 0$ or $y = 1$

When y = 0, x = $\frac{0}{2}$ = 0

When y = 1, x = $\frac{1}{2}$

∴ the points of intersection are O(0, 0) and $\mathrm{B}\left(\frac{1}{2}, 1\right)$

Required area = area of the region OABCO

= area of the region OABDO – area of the region OCBDO

Now, area of the region OABDO

= area under the parabola y^2 = 4ax between x = 0 and x = $\frac{1}{2}$

= $\int_0^{\frac{1}{2}} y \cdot dx, \text{ where } y = \sqrt{2} x$

= $\int_0^{\frac{1}{2}} \sqrt{2} x d x$

= $\sqrt{2}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{\frac{1}{2}}$

= $\sqrt{2}\left[\frac{2}{3}\left(\frac{1}{2}\right)^{\frac{3}{2}}-0\right]$

= $\sqrt{2}\left[\frac{2}{3} \cdot \frac{1}{2 \sqrt{2}}\right]$

= $\frac{1}{3}$

Area of the region OCBDO

= area under the line y

= 4ax between x

= 0 and x = $\frac{1}{4ax}$

$$= \int_0^{\frac{1}{2}} y \cdot dx, \text{where } y = x$$

$$= \int_0^{\frac{1}{2}} 2x \cdot dx$$

$$= \left[\frac{2x^2}{2}\right]_0^{\frac{1}{2}}$$

$$= \frac{4}{3} - 0$$

$$= \frac{2a^2}{1}$$

∴ required area

$$= \frac{4}{3} = \frac{2a^2}{1}$$

$$= \frac{8a^2}{3} \text{sq units.}$$

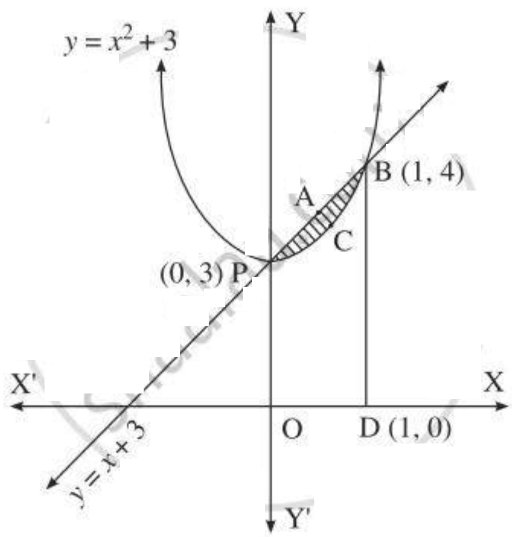
Exercise 5.1 | Q 3.5 | Page 187

Find the area of the region included between: y = x^2 + 3 and the line y = x + 3

SOLUTION

The given parabola is y = x^2 + 3, i.e. $(x – 0)^2$ = y – 3

∴ its vertex is P(0, 3).



To find the points of intersection of the line and the parabola. Equating the values of y from both the equations, we get

$x^3 + 3 = x + 3$
$\therefore x^2 - x = 0$
$\therefore x(x - 1) = 0$
$\therefore x = 0$ or $x = 1$

When $x = 0$, $y = 0 + 3 = 3$
When $x = 1$, $y = 1 + 3 = 4$

$\therefore$ the points of intersection are P(0, 3) and B(1, 4) Required area = area of the region PABCP

= area of the region OPABDO – area of the region OPCBDO
Now, area of the region OPABDO

= area under the line $y = x + 3$ between $x = 0$ and $x = 1$

$$= \int_0^1 y \cdot dx, \text{ where } y = x + 3$$

$$= \int_0^1 (x + 3) \cdot dx$$

$$= \int_0^1 x \cdot dx + 3\int_0^1 1 \cdot dx$$

$= \left[\frac{x^2}{2}\right]_0^1 + 3[x]_0^1$

$= \left(\frac{1}{2} - 0\right) + 3(1 - 0)$

$= \frac{7}{2}$

Area of the region OPCBDO

= area under the parabola y = x^2 + 3 between x = 0 and x = 1

$= \int_0^1 y \cdot dx, \text{ where } y = x^2 + 3$

$= \int_0^1 (x^2 + 3) \cdot dx$

$= \int_0^1 x^2 \cdot dx + 3\int_0^1 1 \cdot dx$

$= \left[\frac{x^3}{3}\right]_0^1 + 3[x]_0^1$

$= \left(\frac{1}{3} - 0\right) + 3(1 - 0)$

$= \frac{10}{3}$

$\therefore$ required area = $\frac{7}{2} - \frac{10}{3}$

$= \frac{21 - 20}{6}$

$= \frac{1}{6} \text{sq unit.}$

MISCELLANEOUS EXERCISE 5 [PAGES 188 - 190]

Miscellaneous Exercise 5 | Q 1.01 | Page 188

Choose the correct option from the given alternatives :

The area bounded by the regional ≤ x ≤ 5 and 2 ≤ y ≤ 5 is given by

1. **12 sq units**
2. 8 sq units
3. 25 sq units
4. 32 sq units

SOLUTION

12 sq units.

Miscellaneous Exercise 5 | Q 1.02 | Page 188

Choose the correct option from the given alternatives :

The area of the region enclosed by the curve $y = \frac{1}{x}$, and the lines $x = e$, $x = e^2$ is given by

1 sq unit

$\frac{1}{2}$ sq unit

$\frac{3}{2}$ sq units

$\frac{5}{2}$ sq units

SOLUTION

1 sq unit.

Miscellaneous Exercise 5 | Q 1.03 | Page 188

Choose the correct option from the given alternatives :

The area bounded by the curve $y = x^3$, the X-axis and the lines $x = -2$ and $x = 1$ is

– 9 sq units

$-\frac{15}{4}$ sq units

$\frac{15}{4}$ sq units

$\frac{17}{4}$ sq units

SOLUTION

$\frac{15}{4}$ sq units.

Miscellaneous Exercise 5 | Q 1.04 | Page 188

Choose the correct option from the given alternatives :

The area enclosed between the parabola $y^2 = 4x$ and line $y = 2x$ is

$\frac{2}{3}$ sq units

$\frac{1}{3}$ sq unit

$\frac{1}{4}$ sq unit

$\frac{3}{4}$ sq unit

SOLUTION

$\frac{1}{3}$ sq unit.

Miscellaneous Exercise 5 | Q 1.05 | Page 188

Choose the correct option from the given alternatives :

The area of the region bounded between the line $x = 4$ and the parabola $y^2 = 16x$ is

$\frac{128}{3}$ sq units

$\frac{108}{3}$ sq units

$\frac{118}{3}$ sq units

$\frac{218}{3}$ sq units

SOLUTION

$\frac{128}{3}$ sq units.

Miscellaneous Exercise 5 | Q 1.06 | Page 189

Choose the correct option from the given alternatives :

The area of the region bounded by y = cos x, Y-axis and the lines x = 0, x = 2π is

1 sq unit

2 sq units

3 sq units

4 sq units

SOLUTION

4 sq units.

Miscellaneous Exercise 5 | Q 1.07 | Page 189

Choose the correct option from the given alternatives :

The area bounded by the parabola $y^2 = 8x$, the X-axis and the latus rectum is

$\frac{31}{3}$ sq units

$\frac{32}{3}$ sq units

$\frac{32\sqrt{2}}{3}$ sq units

$\frac{16}{3}$ sq units

SOLUTION

$\frac{32}{3}$ sq units.

Miscellaneous Exercise 5 | Q 1.08 | Page 189

Choose the correct option from the given alternatives :

The area under the curve y = $2\sqrt{x}$, enclosed between the lines x = 0 and x = 1 is

4 sq units

$\frac{3}{4}$ sq unit

$\frac{2}{3}$ sq unit

$\frac{4}{3}$ sq units

SOLUTION

$\frac{4}{3}$ sq units.

Miscellaneous Exercise 5 | Q 1.09 | Page 189

Choose the correct option from the given alternatives :

The area of the circle $x^2 + y^2 = 25$ in first quadrant is

$\frac{25\pi}{4}$ sq units

5π sq units

5 sq units

3 sq units

SOLUTION

$\frac{25\pi}{4}$ sq units.

Miscellaneous Exercise 5 | Q 1.1 | Page 189

Choose the correct option from the given alternatives :

The area of the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

ab sq units

πab sq units

$\frac{\pi}{\text{ab}}$ sq units

πa^2 sq units

SOLUTION

πab sq units.

Miscellaneous Exercise 5 | Q 1.11 | Page 189

Choose the correct option from the given alternatives :

The area bounded by the parabola $y^2 = x$ and the line $2y = x$ is

- $\frac{4}{3}$ sq unit
- 1 sq unit
- $\frac{2}{3}$ sq unit
- $\frac{1}{3}$ sq unit

SOLUTION

$\frac{4}{3}$ sq unit.

Miscellaneous Exercise 5 | Q 1.12 | Page 189

Choose the correct option from the given alternatives :

The area enclosed between the curve $y = \cos 3x$, $0 \le x \le \frac{\pi}{6}$ and the X-axis is

- $\frac{1}{2}$ sq unit
- 1 sq unit
- $\frac{2}{3}$ sq unit
- $\frac{1}{3}$ sq unit

SOLUTION

$\frac{1}{3}$ sq unit.

Miscellaneous Exercise 5 | Q 1.13 | Page 189

Choose the correct option from the given alternatives :

The area bounded by y = $\sqrt{x}$ and the x = 2y + 3, X-axis in first quadrant is

$2\sqrt{3}$sq units

9 sq units

$\frac{34}{3}$sq units

18 sq units

SOLUTION

9 sq units.

Miscellaneous Exercise 5 | Q 1.14 | Page 189

Choose the correct option from the given alternatives :

The area bounded by the ellipse $\frac{x^2}{a^2}\ \frac{y^2}{b^2}$ = 1 and the line $\frac{x}{a} + \frac{y}{b}$ = 1 is

(πab – 2ab) sq units

$\left(\frac{\pi ab}{4} - \frac{\text{ab}}{2}\right)$sq units

(πab – ab) sq units

πab sq units

SOLUTION

$\left(\frac{\pi ab}{4} - \frac{\text{ab}}{2}\right)$sq units.

Miscellaneous Exercise 5 | Q 1.15 | Page 189

Choose the correct option from the given alternatives :

The area bounded by the parabola y = x^2 and the line y = x is

$\frac{1}{2}$ sq unit
$\frac{1}{3}$ sq unit
$\frac{1}{6}$ sq unit
$\frac{1}{12}$ sq unit

SOLUTION

$\frac{1}{6}$ sq unit.

Miscellaneous Exercise 5 | Q 1.16 | Page 189

Choose the correct option from the given alternatives :

The area enclosed between the two parabolas $y^2 = 4x$ and $y = x$ is

$\frac{16}{3}$ sq units
$\frac{32}{3}$ sq units
$\frac{8}{3}$ sq units
$\frac{4}{3}$ sq units

SOLUTION

$\frac{8}{3}$ sq units.

Miscellaneous Exercise 5 | Q 1.17 | Page 190

Choose the correct option from the given alternatives :

The area bounded by the curve y = tan x, X-axis and the line x = $\frac{\pi}{4}$ is

$\frac{1}{2}\log 2$ sq units
log 2 sq units

2 log 2 sq units

3·log 2 sq units

SOLUTION

$\frac{1}{2}\log 2$ sq units.

Miscellaneous Exercise 5 | Q 1.18 | Page 190

Choose the correct option from the given alternatives :

The area of the region bounded by $x^2 = 16y$, $y = 1$, $y = 4$ and $x = 0$ in the first quadrant, is

$\frac{7}{3}$ sq units

$\frac{8}{3}$ sq units

$\frac{64}{3}$ sq units

$\frac{56}{3}$ sq units

SOLUTION

$\frac{56}{3}$ sq units.

Miscellaneous Exercise 5 | Q 1.19 | Page 190

Choose the correct option from the given alternatives :

The area of the region included between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, $(a > 0)$ is given by

$\frac{16a^2}{3}$ sq units

$\frac{8a^2}{3}$ sq units

$\frac{64}{3}$ sq units

$\frac{56}{3}$ sq units

SOLUTION

$\frac{16a^2}{3}$ sq units.

Miscellaneous Exercise 5 | Q 1.2 | Page 190

Choose the correct option from the given alternatives :

The area of the region included between the line x + y = 1 and the circle $x^2 + y^2 = 1$ is

$\left(\frac{\pi}{2} - 1\right)$ sq units

$(\pi - 2)$ sq units

$\left(\frac{\pi}{4} - \frac{1}{2}\right)$ sq units

$\left(\pi - \frac{1}{2}\right)$ sq units

SOLUTION

$\left(\frac{\pi}{4} - \frac{1}{2}\right)$ sq units

Miscellaneous Exercise 5 | Q 2.01 | Page 190

Solve the following :
Find the area of the region bounded by the following curve, the X-axis and the given lines : $0 \le x \le 5$, $0 \le y \le 2$

SOLUTION

Required area = $\int_0^5 y \cdot dx$, where $y = 2$

$= \int_0^5 2 \cdot dx = [2x]_0^5$

= 2 x 5 – 0

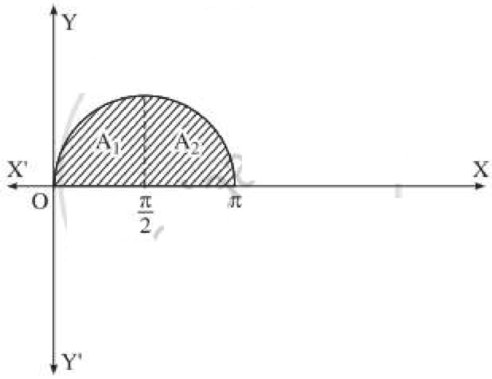
= 10 sq units.

Miscellaneous Exercise 5 | Q 2.01 | Page 190

Solve the following :
Find the area of the region bounded by the following curve, the X-axis and the given lines : y = sin x, x = 0, x = π

SOLUTION

The curve $y = \sin x$ intersects the X-axis at $x = 0$ and $x = \pi$ between $x = 0$ and $x = \pi$.



Two bounded regions A_1 and A_2 are obtained. Both the regions have equal areas.
∴ required area = $A_1 + A_2 = 2A_1$

$$= 2\int_0^{\frac{\pi}{2}} y \cdot dx, \text{ where } y = \sin x$$

$$= 2\int_0^{\frac{\pi}{2}} \sin x \cdot dx$$

$$= 2[-\cos x]_0^{\frac{\pi}{2}}$$

$$= 2\left[-\cos\ \frac{\pi}{2}\ \cos 0\right]$$

$$= 2(-0 + 1)$$

$= 2$ sq units.

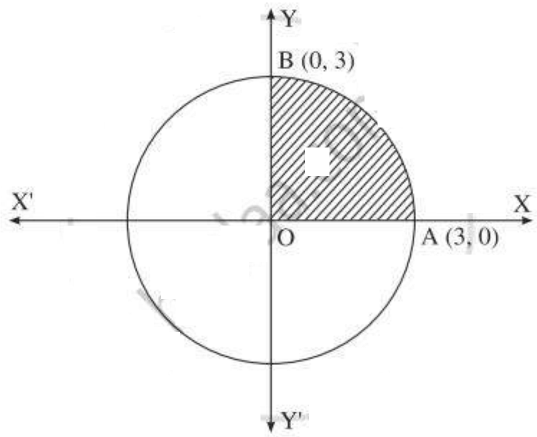
Miscellaneous Exercise 5 | Q 2.02 | Page 190

Solve the following :

Find the area of the circle $x^2 + y^2 = 9$, using integration.

SOLUTION

By the symmetry of the circle, its area is equal to 4 times the area of the region OABO. Clearly for this region, the limits of integration are 0 and 3.



From the equation of the circle, y2 = 9 – x^2.

In the first quadrant, y > 0

$\therefore$ y = $\sqrt{9-x^2}$

$\therefore$ area of the circle = 4 (area of the region OABO)

= $4\int_0^3 y \cdot dx = 4\int_0^3 \sqrt{9-x^2} \cdot dx$

= $4\left[\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3$

= $4\left[\frac{3}{2}\sqrt{9-9}+\frac{9}{2}\sin^{-1}\left(\frac{3}{3}\right)\right]-4\left[\frac{0}{2}\sqrt{9-0}+\frac{9}{2}\sin^{1}(0)\right]$

= $4 \cdot \frac{9}{2} \cdot \frac{\pi}{2}$

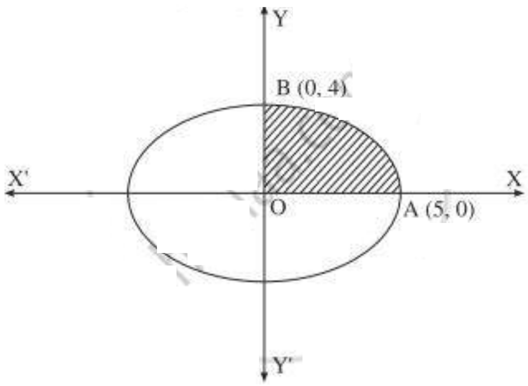
= 9π sq units.

Miscellaneous Exercise 5 | Q 2.03 | Page 190

Solve the following :

Find the area of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}$ = 1 using integration

SOLUTION



By the symmetry of the ellipse, its area is equal to 4 times the area of the region OABO. Clearly for this region, the limits of integration are 0 and 5.

From the equation of the ellipse

$$\frac{y^2}{16} = 1 - \frac{x^2}{25} = \frac{25 - x^2}{25}$$

$\therefore y^2 = \frac{16}{25}\left(25 - x^2\right)$

In the first quadrant y > 0

$\therefore y = \frac{4}{5}\sqrt{25 - x^2}$

$\therefore$ area of the ellipse = 4 (area of the region OABO)

$$= 4\int_0^5 y \cdot dx$$

$$= \int_0^5 \frac{4}{5}\sqrt{25 - x^2} \cdot dx$$

$$= \frac{16}{5}\int_0^5 \sqrt{25 - x^2} \cdot dx$$

$$= \frac{16}{5}\left[\frac{x}{2}\sqrt{25 - x^2} + \frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right)\right]_0^5$$

$$= \frac{16}{5}\left(\frac{5}{2}\sqrt{25 - 25} + \frac{25}{2}\sin^{-1}(1)\right) - \frac{16}{5}\left[\frac{5}{2}\sqrt{25 - 0} + \frac{25}{2}\sin^{-1}(0)\right]$$

$$= \frac{16}{5} \times \frac{25}{2} \times \frac{\pi}{2}$$

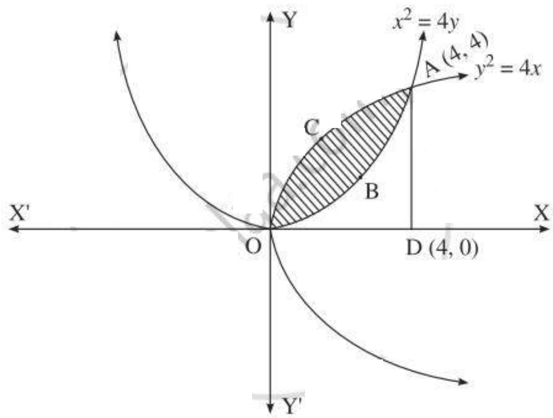
$= 20\pi$ sq units.

Miscellaneous Exercise 5 | Q 2.04 | Page 190

Solve the following :

Find the area of the region lying between the parabolas :
$y^2 = 4x$ and $x^2 = 4y$

SOLUTION



For finding the points of intersection of the two parabolas, we equate the values of y^2 from their equations.

From the equation $x^2 = 4y$, $y = \frac{x^2}{4}$

$\therefore y = \frac{x^4}{16}$

$\therefore \frac{x^4}{16} = 4x$

$\therefore x^4 - 64x = 0$

$\therefore x(x^3 - 64) = 0$

$\therefore x = 0$ or $x^3 = 64$

i.e. $x = 0$ or $x = 4$

When $x = 0$, $y = 0$

When x = 4, y = $\frac{4^2}{4}$ = 4

∴ the points of intersection are O(0, 0) and A(4, 4).

Required area = area of the region OBACO

= [area of the region ODACO] − [area of the region ODABO]

Now, area of the region ODACO

= area under the parabola $y^2 = 4x$,

i.e. y = $2\sqrt{x}$ between x = 0 and x = 4

= $\int_0^4 2\sqrt{x} \cdot dx$

= $\left[2\,\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$

= $2 \times \frac{2}{3} \times 4^{\frac{3}{2}} - 0$

= $\frac{4}{3} \times \left(2^3\right)$

= $\frac{32}{3}$

Area ofthe region ODABO

= area under the rabola $x^2 = 4y$,

i.e. y = $\frac{x^2}{4}$ between x = 0 and x = 4

= $\int_0^4 \frac{1}{4} x^2 \cdot dx$

$$= \frac{1}{4}\left[\frac{x^3}{3}\right]_0^4$$

$$= \frac{1}{4}\left(\frac{64}{3} - 0\right)$$

$$= \frac{16}{3}$$

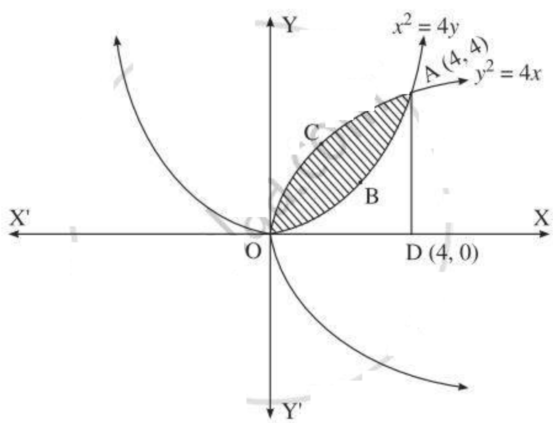
$\therefore$ required area $= \frac{32}{3} - \frac{16}{3}$

$$= \frac{16}{3} \text{ sq units.}$$

Miscellaneous Exercise 5 | Q 2.04 | Page 190

Solve the following :
Find the area of the region lying between the parabolas : $y^2 = x$ and $x^2 = y$.

SOLUTION



For finding the points of intersection of the two parabolas, we equate the values of y^2 from their equations.

From the equation x^2 = y, y = $\frac{x^2}{y}$

∴ y = $\frac{x^2}{y}$

∴ $\frac{x^2}{y}$ = x

∴ $x^2 - y = 0$

∴ $x(x^3 - y) = 0$

∴ x = 0 or x^3 = y

i.e. x = 0 or x = 4

When x = 0, y = 0

When x = 4, y = $\frac{4^2}{4}$ = 4

∴ the points of intersection are O(0, 0) and A(4, 4).

Required area = area of the region OBACO

= [area of the region ODACO] – [area of the region ODABO]

Now, area of the region ODACO

= area under the parabola y^2 = 4x,

i.e. y = $2\sqrt{x}$ between x = 0 and x = 4

$$= \int_0^4 2\sqrt{x} \cdot dx$$

$$= \left[2\,\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$$

$$= 2 \times \frac{2}{3} \times 4^{\frac{3}{2}} - 0$$

$$= \frac{4}{3} \times \left(2^3\right)$$

$= \frac{32}{3}$

Area ofthe region ODABO

= area under the rabola $x^2 = 4y$,

i.e. y = $\frac{x^2}{3}$ between x = 0 and x = 4

$= \int_0^4 \frac{1}{3} x^2 \cdot dx$

$= \frac{1}{3}\left[\frac{x}{3}\right]_0^4$

$= \frac{1}{3}\left(\frac{y}{3} - 0\right)$

$= \frac{y}{3}$

∴ required area = $\frac{1}{3} - \frac{y}{3}$

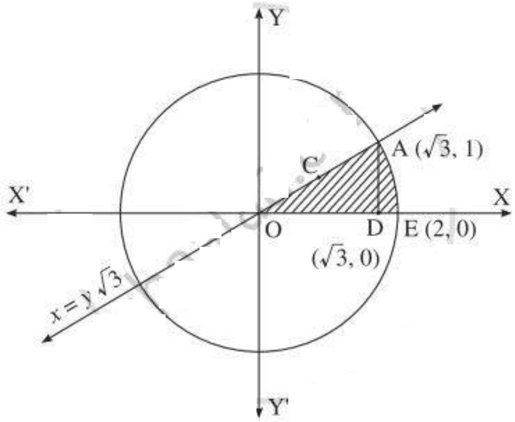
$= \frac{1}{3}$ sq units.

Miscellaneous Exercise 5 | Q 2.05 | Page 190

Solve the following :

Find the area of the region in first quadrant bounded by the circle $x^2 + y^2 = 4$ and the X-axis and the line x = y √3.

SOLUTION



For finding the point of intersection of the circle and the line, we solve

$x^2 + y^2 = 4$ ...(1)

and x = $y\sqrt{3}$ ...(2)

From (2), $x^2 = 3y$

From (1), $x^2 = 4 - y^2$

$\therefore\ 3y^2 = 4 - y^2$

$\therefore 4y^2 = 4$

$\therefore\ y^2 = 1$

$\therefore$ y = 1 in the first quadrant.

When y = , x = 1 x $\sqrt{3} = \sqrt{3}$

$\therefore$ the circle and the line intersect at $\mathbf{A}\left(\sqrt{3}, 1\right)$ in the first quadrant

Required area = area of the region OCAEDO

= area of the region OCADO + area of the region DAED

Now, area of the region OCADO

= area under the line x $y\sqrt{3}$

i.e. y = $\frac{x}{\sqrt{y}}$ between x = 0 and x = $\sqrt{3}$

$$= \int_0^{\sqrt{3}} \frac{x}{\sqrt{3}} \cdot dx$$

$$= \left[\frac{x^2}{2\sqrt{3}}\right]_0^{\sqrt{3}}$$

$$= \frac{3}{2\sqrt{3}} - 0$$

$$= \frac{\sqrt{3}}{2}$$

Area of the region DAED

= area under the circle $x^2 + y^2 = 4$ i.e. $y = +\sqrt{4 - x^2}$ (in the first quadrant) between $x = \sqrt{3}$ and $x = 2$

$$= \int_{\sqrt{3}}^{2} \sqrt{4 - x^2} \cdot dx$$

$$= \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^{2}$$

$$= \left[\frac{2}{2}\sqrt{4 - 4} + 2\sin^{-1}(1)\right] - \left[\frac{\sqrt{3}}{2}\sqrt{4 - 3} + 2\sin^{-1}\ \frac{\sqrt{3}}{2}\right]$$

$$= 0 + 2\left(\frac{\pi}{2}\right) - \frac{\sqrt{3}}{2} - 2\left(\frac{\pi}{3}\right)$$

$$= \pi - \frac{\sqrt{3}}{2} - \frac{2\pi}{3}$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{2}$$

$\therefore$ required area = $\frac{\sqrt{3}}{2} + \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$

$= \frac{\pi}{3}$ sq units.

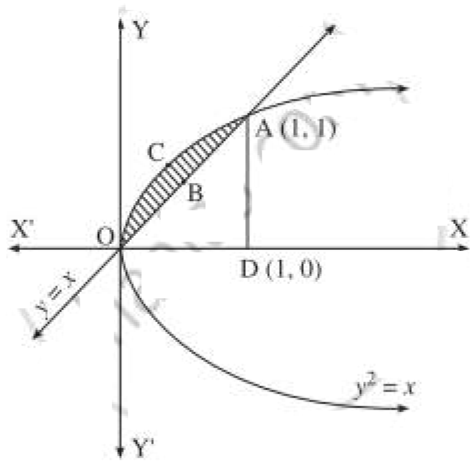
Miscellaneous Exercise 5 | Q 2.06 | Page 190

Solve the following :

Find the area of the region bounded by the parabola $y^2 = x$ and the line $y = x$ in the first quadrant.

SOLUTION

To obtain the points of intersection of the line and the parabola, we equate the values of x from both the equations.



∴ $y^2 = y$
∴ $y^2 - y = 0$
∴ $y(y - 1) = 0$
∴ $y = 0$ or $y = 1$

When $y = 0$, $x = 0$
When $y = 1$, $x = 1$

∴ the points of intersection are O(0, 0) and A(1, 1). Required area of the region OCABO = area of the region OCADO – area of the region OBADO

Now, area of the region OCADO

= area under the parabola $y^2 = x$ i.e. $y = \pm\sqrt{x}$ (in the first quadrant) between $x = 0$ and $x = 1$

$$= \int_0^1 \sqrt{x} \cdot dx$$

$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1$$

$$= \frac{2}{3} \times (1 - 0)$$

$$= \frac{2}{3}$$

Area of the region OBADO

= area under the line y = x between x 0 and x = 1

$$= \int_0^1 x \cdot dx$$

$$= \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{2} - 0$$

$$= \frac{2}{3}$$

$$\therefore \text{ required area } = \frac{2}{3} - \frac{1}{2}$$

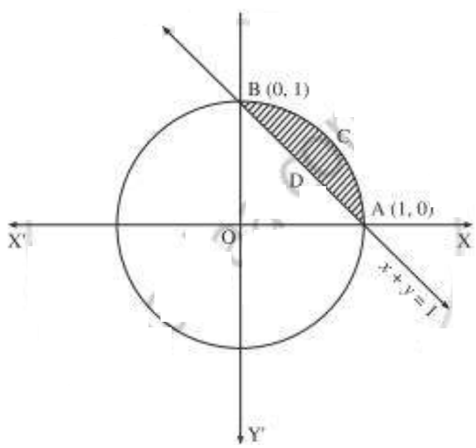
$$= \frac{1}{6} \text{sq unit.}$$

Miscellaneous Exercise 5 | Q 2.07 | Page 190

Solve the following :

Find the area enclosed between the circle $x^2 + y^2 = 1$ and the line $x + y = 1$, lying in the first quadrant.

SOLUTION



Required area = area of the region ACBDA
= (area of the region OACBO) – (area of the region OADBO)
Now, area of the region OACBO
= area under the circle $x^2 + y^2 = 1$ between x = 0 and x = 1

$= \int_0^1 \cdot dx,\ \text{where } y^2 = 1 - x^2,$

i.e. y = $\sqrt{1 - x^2}$, as y > 0

$= \int_0^1 \sqrt{1 - x^2} \cdot dx$

$= \left[\frac{x}{2}\sqrt{1 - x^2} + \frac{1}{2}\sin^{-1}(x)\right]_0^1$

$= \frac{1}{2}\sqrt{1 - 1} + \frac{1}{2}\sin^1 1 - 0$

$= \frac{1}{2} \times \frac{\pi}{2}$

$= \frac{\pi}{4}$

Area of the region OADBO

= area under the line x + y = 1 between x = 0 and x = 1

$= \int_0^1 y \cdot dx,\ \text{where } y = 1 - x$

$= \int_0^1 (1 - x) \cdot dx$

$= \left[x - \frac{x^2}{2}\right]_0^1$

$= 1 - \frac{1}{2} - 0$

$= \frac{1}{2}$

∴ required area = $\left(\frac{\pi}{4} - \frac{1}{2}\right)\text{sq units.}$

Miscellaneous Exercise 5 | Q 2.08 | Page 190

Solve the following :

Find the area of the region bounded by the curve $(y – 1)^2 = 4(x + 1)$ and the line y = (x – 1).

SOLUTION

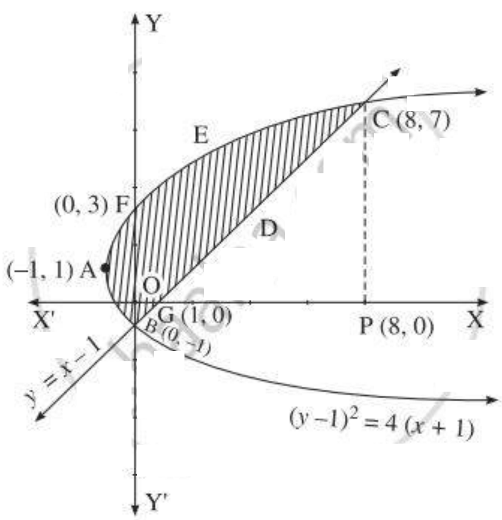
The equation of the curve is $(y – 1)^2 = 4(x + 1)$
This is a parabola with vertex at A(– 1, 1).

To find the points of intersection of the line y = x – 1 and the parabola.
Put y = x – 1 in the equation of the parabola, we get
$(x – 1 – 1)^2 = 4(x + 1)$

$\therefore x^2 – 4x + 4 = 4x + 4$
$\therefore x^2 – 8x = 0$
$\therefore x(x – 8) = 0$

$\therefore$ x = 0, x = 8
When x = 0, y = 0 – 1 = – 1
When x = 8, y = 8 – 1 = 7

$\therefore$ the points of intersection are B(0, – 1) and C(8, 7)



To find the points where the parabola (y – 1)2 = 4(x + 1) cuts the Y-axis.
Put x = 0 in the equation of the parabola, we get

(y – 1)2 = 4(0 + 1) = 4
$\therefore$ y – 1 = ± 2
$\therefore$ y – 1 =2 or y – 1 = – 2

$\therefore$ y = 3 or y = – 1
$\therefore$ the parabola cuts the Y-axis at the points B (0, – 1) and F(0, 3).

To find the point where the line y = x – 1 cuts the X-axis. Put y = 0 in the equation of the line, we get

x – 1 = 0
∴ x = 1
∴ the line cuts the X-axis at the point G (1, 0).

Required area = area of the region BFAB + area of the region OGDCEFO + area of the region OBGO

Now, area of the region BFAB
= area under the parabola $(y - 1)^2 = 4(x + 1)$, Y-axis from y = – 1 to y = 3

$$= \int_{-1}^{3} x \cdot dy,\ \text{where } x + 1 = \frac{(y-1)^2}{4}, \text{i.e.} x = \frac{(y-1)^2}{4} - 1$$

$$= \int_{-1}^{3} \left[\frac{(y-1)^2}{4} - 1\right] \cdot dy$$

$$= \left[\frac{1}{4} \cdot \frac{(y-1)^3}{3} - y\right]_{-1}^{3}$$

$$= \left[\left\{\frac{1}{12}(3-1)^3 - 3\right\} - \left\{\frac{1}{12}(-1-1)^3 - (-1)\right\}\right]$$

$$= \frac{8}{12} - 3 + \frac{8}{12} - 1$$

$$= \frac{16}{12} - 4$$

$$= \frac{4}{3} - 4$$

$$= -\frac{8}{3}$$

Since, area cannot be negative, area of the region BFAB

$= \left| -\frac{8}{3} \right|$

$= \frac{8}{3}\text{sq units.}$

Area of the region OGDCEFO

= area of the region OPCEFO – area of the region GPCDG

$= \int_0^8 y \cdot dx,\ \text{where } (y-1)^2$

$= 4(x+1), \text{i.e.} y = 2\sqrt{x+1} + 1 - \int_1^8 y \cdot dx,\ \text{where } y = x - 1$

$= \int_0^8 \left[2\sqrt{x+1} + 1\right] \cdot dx - \int_1^8 (x-1) \cdot dx$

$= \left[\frac{2 \cdot (x+1)^{\frac{3}{2}}}{\frac{3}{2}} + x\right]_0^8 - \left[\frac{x^2}{2} - x\right]_1^8$

$= \left[\frac{4}{3}(9)^{\frac{3}{2}} + 8 - \frac{4}{3}(1)^{\frac{3}{2}} - 0\right] - \left[\left(\frac{64}{2} - 8\right) - \left(\frac{1}{2} - 1\right)\right]$

$= \left(36 + 8 - \frac{4}{3}\right) - \left(24 + \frac{1}{2}\right)$

$= 44 - \frac{4}{3} - 24 - \frac{1}{2}$

$= 20 - \left(\frac{4}{3} + \frac{1}{2}\right)$

$= 20 - \frac{11}{6}$

$= \frac{109}{6}\text{sq units.}$

Area of region OBGO = $\int_0^1 y \cdot dx,\ \text{where } y = x - 1$

$= \int_0^1 (x-1) \cdot dx$

$= \left[\frac{x^2}{2} - x\right]_0^1$

$= \frac{1}{2} - 1 - 0$

$= -\frac{1}{2}$

Since, area cannot be negative,

area of the region = $\left|-\frac{1}{2}\right| = \frac{1}{2}\text{sq unit.}$

$\therefore$ required area = $\frac{8}{3} + \frac{109}{6} + \frac{1}{2}$

$= \frac{16 + 109 + 3}{6}$

$= \frac{128}{6}$

$= \frac{64}{3}\text{sq units.}$

Miscellaneous Exercise 5 | Q 2.09 | Page 190

Solve the following :
Find the area of the region bounded by the straight line 2y = 5x + 7, X-axis and x = 2, x = 5.

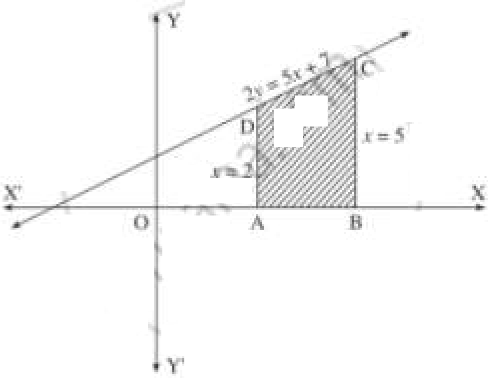
SOLUTION

The equation of the line is 2y = 5x+ 7,

i.e., y = $\frac{5}{2}x + \frac{7}{2}$

Required area = area of the region ABCDA

= area under the line y = $5\frac{)}{2}x + \frac{7}{2}$ between x = 2 and x = 5



$$= \int_2^5 \left(\frac{5}{2}x + \frac{7}{2}\right) \cdot dx$$

$$= \frac{5}{2} \cdot \int_2^5 x \cdot dx + \frac{7}{2}\int_2^5 1 \cdot dx$$

$$= \frac{5}{2}\left[\frac{x^2}{2}\right]_2^5 + \frac{7}{2}[x]_2^5$$

$$= \frac{5}{2}\left[\frac{25}{2} - \frac{4}{2}\right] + \frac{7}{2}[5-2]$$

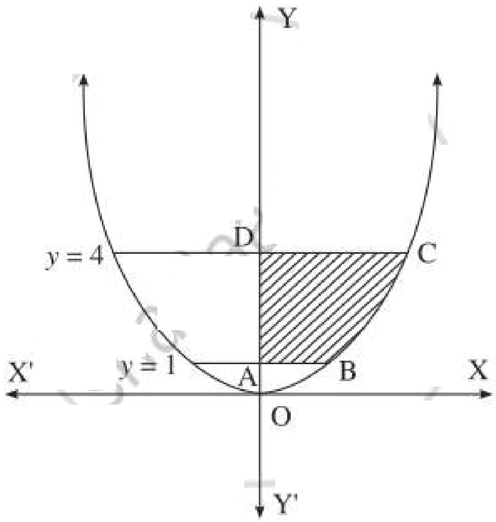
$$= \frac{5}{2} \times \frac{21}{2} + \frac{21}{2}$$

$$= \frac{105}{4} + \frac{42}{4}$$

$$= \frac{147}{4} \text{sq units.}$$

Miscellaneous Exercise 5 | Q 2.10 | Page 190

Solve the following :
Find the area of the region bounded by the curve $y = 4x^2$, Y-axis and the lines y = 1, y = 4.
SOLUTION



By symmetry of the parabola, the required area is 2 times the area of the region ABCD.

From the equation of the parabola, $x^2 = \frac{y}{4}$

the first quadrant, x > 0

$\therefore x = \frac{1}{2}\sqrt{y}$

$\therefore$ required area = $\int_1^4 x \cdot dy$

$= \frac{1}{2}\int_1^4 \sqrt{y} \cdot dy$

$= \frac{1}{2}\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4$

$= \frac{1}{2} \times \frac{2}{3}\left[4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]$

$= \frac{1}{3}\left[\left(2^2\right)^{\frac{3}{2}} - 1\right]$

$= \frac{1}{3}[8 - 1]$

$= \frac{7}{3}\text{sq units.}$